Lecture 30. Orthogonal sets

Def A set of nonzero vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_m \in \mathbb{R}^n$ is orthogonal if each pair from the set is orthogonal (i.e., $\vec{v}_i \cdot \vec{v}_j = 0$ for $i \neq j$)

e.g. the standard basis of $\mathbb{R}^n$ given by $\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n$.

Note It is very important to check the orthogonality of every pair.

e.g. $\vec{v}_1 = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

$$\Rightarrow \left| \begin{array}{l} \vec{v}_1 \cdot \vec{v}_2 = 1\cdot 2 + 4 \cdot 1 + (-3)\cdot 2 = 0 \\ \vec{v}_1 \cdot \vec{v}_3 = 1 \cdot 1 + 4 \cdot (-1) + (-3)\cdot(-1) = 0 \\ \vec{v}_2 \cdot \vec{v}_3 = 2 \cdot 1 + 1 \cdot (-1) + 2 \cdot (-1) = -1 \end{array} \right.$$

$\Rightarrow \vec{v}_1, \vec{v}_2, \vec{v}_3$ are not orthogonal

Prop Orthogonal vectors are linearly independent.

pf Let $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_m$ be orthogonal vectors in $\mathbb{R}^n$.

Suppose $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m = \vec{0}$ with $c_0, c_1, \cdots, c_m \in \mathbb{R}$.

$\Rightarrow \vec{v}_i \cdot (c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m) = \vec{v}_i \cdot \vec{0}$

$\Rightarrow c_1\vec{v}_i \cdot \vec{v}_1 + c_2\vec{v}_i \cdot \vec{v}_2 + \cdots + c_m\vec{v}_i \cdot \vec{v}_m = 0$

$\Rightarrow c_i\vec{v}_i \cdot \vec{v}_i = 0$ ($\vec{v}_i \cdot \vec{v}_j = 0$ for $i \neq j$)

$\Rightarrow c_i \|\vec{v}_i\|^2 = 0$

$\Rightarrow c_i = 0$ (each $\vec{v}_i$ is nonzero)

Hence $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_m$ are linearly independent.

Prop If vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n \in \mathbb{R}^n$ are orthogonal, they form a basis.

pf Take A to be the matrix with columns $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$.

Since $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ are orthogonal, they are linearly independent.

$\Rightarrow$ RREF(A) has a leading 1 in every column

$\Rightarrow$ RREF(A) $= I$ (A is a square matrix)

$\Rightarrow$ $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ form a basis of $\mathbb{R}^n$

Note Most bases of $\mathbb{R}^n$ are not orthogonal

Prop Given an orthogonal basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ of $\mathbb{R}^n$, the $\mathcal{B}$-coordinate vector of $\vec{v} \in \mathbb{R}^n$ is

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \text{ with } c_i = \frac{\vec{v} \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i}.$$

pf We have $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$

$\Rightarrow$ $\vec{v} \cdot \vec{v}_i = (c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n) \cdot \vec{v}_i$

$\Rightarrow$ $\vec{v} \cdot \vec{v}_i = c_1\vec{v}_1 \cdot \vec{v}_i + c_2\vec{v}_2 \cdot \vec{v}_i + \cdots + c_n\vec{v}_n \cdot \vec{v}_i$

$\Rightarrow$ $\vec{v} \cdot \vec{v}_i = c_i\vec{v}_i \cdot \vec{v}_i$ $(\vec{v}_i \cdot \vec{v}_j = 0 \text{ for } i \neq j)$

$\Rightarrow$ $c_i = \dfrac{\vec{v} \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i}$

Ex Consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 4 \\ 1 \\ -7 \end{bmatrix}$$

(1) Determine whether $\vec{v}_1, \vec{v}_2, \vec{v}_3$ form an orthogonal basis of $\mathbb{R}^3$.

Sol We have

$$\begin{cases} \vec{v}_1 \cdot \vec{v}_2 = 1 \cdot 2 + 3 \cdot (-1) + 1 \cdot 1 = 0 \\ \vec{v}_1 \cdot \vec{v}_3 = 1 \cdot 4 + 3 \cdot 1 + 1 \cdot (-7) = 0 \\ \vec{v}_2 \cdot \vec{v}_3 = 2 \cdot 4 + (-1) \cdot 1 + 1 \cdot (-7) = 0 \end{cases}$$

$\Longrightarrow$ $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are orthogonal

$\Longrightarrow$ $\vec{v}_1, \vec{v}_2, \vec{v}_3$ form $\boxed{\text{an orthogonal basis of } \mathbb{R}^3}$

(2) If possible, express the vector

$$\vec{w} = \begin{bmatrix} -8 \\ 8 \\ 6 \end{bmatrix}$$

as a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

Sol $\vec{v}_1, \vec{v}_2, \vec{v}_3$ form an orthogonal basis of $\mathbb{R}^3$.

$\Longrightarrow$ $\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$ with

$$c_1 = \frac{\vec{w} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} = \frac{(-8) \cdot 1 + 8 \cdot 3 + 6 \cdot 1}{1^2 + 3^2 + 1^2} = \frac{22}{11} = 2,$$

$$c_2 = \frac{\vec{w} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} = \frac{(-8) \cdot 2 + 8 \cdot (-1) + 6 \cdot 1}{2^2 + (-1)^2 + 1^2} = \frac{-18}{6} = -3,$$

$$c_3 = \frac{\vec{w} \cdot \vec{v}_3}{\vec{v}_3 \cdot \vec{v}_3} = \frac{(-8) \cdot 4 + 8 \cdot 1 + 6 \cdot (-7)}{4^2 + 1^2 + (-7)^2} = \frac{-66}{66} = -1.$$

$\Longrightarrow$ $\vec{w} = \boxed{2\vec{v}_1 - 3\vec{v}_2 - \vec{v}_3}$